

# Promiscuously Quadratic Rings

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## Abstract

We register, explicitly, equivalences and dual equivalences between categories of abstract quadratic forms theories and (sub)categories of multi-rings and multifields.

## 1 Basic Concepts

This section is a compiled o basic definitions and results included for the convenience of the reader. For more details and examples, consult [3], [2], [1] or [4].

**Definition 1.1** A multigroup is a quadruple  $(G, *, r, 1)$ , where  $G$  is a non-empty set,

$*$  :  $G \times G \rightarrow \mathcal{P}(G) \setminus \{\emptyset\}$  and  $r : G \rightarrow G$  are functions, and  $1$  is an element of  $G$  satisfying:

- i - If  $z \in x * y$  then  $x \in z * r(y)$  and  $y \in r(x) * z$ .
- ii -  $y \in 1 * x$  iff  $x = y$ .
- iii - With the convention  $x * (y * z) = \bigcup_{w \in y * z} x * w$  and  $(x * y) * z = \bigcup_{t \in x * y} t * z$ ,

$$x * (y * z) = (x * y) * z \text{ for all } x, y, z \in G.$$

A multigroup is said to be *commutative* if

- iv -  $x * y = y * x$  for all  $x, y \in G$ .

**Definition 1.2** A multiring is a sextuple  $(R, +, \cdot, -, 0, 1)$  where  $R$  is a non-empty set,  $+$  :  $R \times R \rightarrow \mathcal{P}(R) \setminus \{\emptyset\}$ ,  $\cdot$  :  $R \times R \rightarrow R$  and  $-$  :  $R \rightarrow R$  are functions,  $0$  and  $1$  are elements of  $R$  satisfying:

- i -  $(R, +, -, 0)$  is a commutative multigroup;

- ii -  $(R, \cdot, 1)$  is a commutative monoid;
- iii -  $a0 = 0$  for all  $a \in R$ ;
- iv - If  $c \in a + b$ , then  $cd \in ad + bd$ . Or equivalently,  $(a + b)d \subseteq ad + bd$ .

$R$  is said to be a multidomain if do not have zero divisors, and  $R$  will be a multifield if every non-zero element of  $R$  has multiplicative inverse. If  $(a + b)d = ad + bd$  for all  $a, b, d \in R$ , then  $R$  will be an *hyperring*.

**Example 1.3**

- a - Every ring, domain and field is a multiring, multidomain and multifield respectively.
- b -  $Q_2 = \{-1, 0, 1\}$  is a multifield with the usual product and the multivalued sum defined by relations

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1, (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

- c - In the set  $\mathbb{R}_+$  of positive real numbers, we define  $a \nabla b = \{c \in \mathbb{R}_+ : |a - b| \leq c \leq a + b\}$ . We have that  $\mathbb{R}_+$  with the usual product and  $\nabla$  multivalued sum is a multifield, called triangle multifield [5]. We denote this multifield by  $\mathcal{T}\mathbb{R}_+$ .

**Lemma 1.4** *Let  $F$  be a multifield. Then  $(a+b)d = ad+bd$  for every  $a, b, d \in F$ .*

**Proof:**

We have  $(a + b)d \subseteq ad + bd$  already. For the other inclusion, if  $d = 0$ , it is done. If  $d \neq 0$ , we have:

$$\begin{aligned} (ad + bd)d^{-1} &\subseteq (ad)d^{-1} + (bd)d^{-1} = ad + bd \Rightarrow \\ ad + bd &= [(ad + bd)d^{-1}]d \subseteq (a + b)d. \end{aligned}$$

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Ideals, spectrum, orderings and another constructions in the multivalued language can be found in [3].

**Definition 1.5** Let  $A$  and  $B$  multirings. A map  $f : A \rightarrow B$  is a morphism if for all  $a, b, c \in A$ :

- i -  $c \in a + b \Rightarrow f(c) \in f(a) + f(b)$ ;
- ii -  $f(-a) = -f(a)$ ;
- iii -  $f(0) = 0$ ;
- iv -  $f(ab) = f(a)f(b)$ ;
- v -  $f(1) = 1$ .

For multirings, there are various sorts of “substructure” that one can consider. If  $A, B$  are multirings, we say  $A$  is embedded in  $B$  by the morphism  $\iota : A \rightarrow B$  if  $\iota$  is injective. We say  $A$  is strongly embedded in  $B$  if  $A$  is embedded in  $B$  and, for all  $a, b, c \in A$ ,  $\iota(c) \in \iota(a) +_B \iota(b) \Rightarrow c \in a +_A b$ . We say  $A$  is a submultiring of  $B$  if  $A$  is strongly embedded in  $B$  and, for all  $a, b \in A$  and all  $c \in B$ ,  $c \in \iota(a) +_B \iota(b) \Rightarrow c \in \iota(A)$ . Note that in the rings case, these all definitions coincide.

The category of multifields (respectively multirings) and their morphisms will be denoted by  $\mathcal{MF}$  (respectively  $\mathcal{MR}$ ). Now, we present the main construction:

**Definition 1.6 (Marshall’s Quotient)** Fix a multiring  $A$  and a multiplicative subset  $S$  of  $A$ . Define an equivalence relation  $\sim$  on  $A$  by  $a \sim b$  iff  $as = bt$  for some  $s, t \in S$ . Denote by  $\bar{a}$  the equivalence class of  $a$  and set  $A/_m S = \{\bar{a} : a \in A\}$ . Defining  $\bar{a} + \bar{b} = \{\bar{c} : cv \in as + bt, \text{ for some } s, t, v \in S\}$ ,  $-\bar{a} = \overline{-a}$ , and  $\bar{a}\bar{b} = \overline{ab}$  we have that  $(A/_m S, +, \cdot, -, \bar{0}, \bar{1})$  is a multiring. When  $S = \sum A^{*2}$ , we will denote  $A/_m \sum A^{*2} = Q_{\text{red}(A)}$ .

## 2 Promiscuously Quadratic Ring

Now, our interesting is analyze the Marshall’s quotient in the case  $(A, S)$  where  $A$  is a commutative ring. Our subject is creat a new axiomatic for abstract quadratic forms, correlated to the theory of special groups [2] and realsemigroups [1]. Let start with an example:

**Example 2.1** Let  $A = \mathbb{Z}/_m(\mathbb{Z}^2 \setminus \{0\})$  and  $B = \mathbb{Q}/_m(\mathbb{Q}^2 \setminus \{0\})$ . Then  $\bar{a} \in \bar{b} + \bar{c}$  (in  $A$ ) if there exist  $r, s, t \in \mathbb{Z}^*$  such that  $ar^2 = bs^2 + ct^2$ . As  $r \neq 0$ ,

$$a = b \cdot \frac{s^2}{r^2} + c \cdot \frac{t^2}{r^2}$$

in  $\mathbb{Q}$ . Therefore  $\bar{a} \in \bar{b} + \bar{c}$  in  $A$  if and only if  $\bar{a} \in \bar{b} + \bar{c}$  in  $B$ . Hence  $B = M(G(\mathbb{Q}))$ , we have that  $A$  is a pre-special subgroup of  $G(\mathbb{Q})$ .

Finally,  $A = \{\bar{p} : p \text{ is free of squares in } \mathbb{Z}\}$ .

Motivated by this example, we developed this definition:

**Definition 2.2 (Promiscuously Quadratic Ring)** A *promiscuously quadratic ring* (*pq-ring*) is a pair  $(A, S)$  where  $A$  is a commutative ring,  $S \subseteq A$ , satisfying the following properties:

**PQ1** -  $1 \in S$ ;

**PQ2** -  $S \cdot S \subseteq S$ ;

**PQ3** - For all  $a \in A$ , there exist  $r, s \in S$  such that  $a^3 r = as$ ;

**PQ4** - For all  $a, b \in A$ , there exist  $r, s, t \in S$  such that  $br = as - at$

**PQ5** - For all  $a, b, c, d \in A$ , if there exists  $u, v, r_1, r_2, s_1, s_2, t_1, t_2 \in S$  such that  $abu = cdv$ ,  $ar_1 = cs_1 + dt_1$  and  $br_2 = cs_2 + dt_2$ , then for all  $x \in A$ , if there exist  $r_3, s_3, t_3 \in S$  such that  $xr_3 = as_3 + bt_3$ , then there exist  $r_4, s_4, t_4 \in S$  such that  $xr_4 = cs_4 + bt_4$ .

$(A, S)$  will be a *reduced promiscuously quadratic ring (rpq-ring)* if  $S + S \subseteq S$ . In this case,  $q(A) = A/mS$  will be the *quadratic multiring* associated to  $(A, S)$ .

Note that [PQ3] state that  $\bar{x}^3 = \bar{x}$  for all  $\bar{x}$  in  $q(A)$  and [PQ4] that  $\bar{a} + (-\bar{a}) = q(A)$  for all  $\bar{a} \in q(A)$  (and this implies  $\bar{a} \in \bar{a} + \bar{b}$  for all  $\bar{a}, \bar{b} \in q(A)$ ). [PQ5] states that For all  $a, b, c, d \in A$ , if  $\overline{ab} = \overline{cd}$  and  $\bar{a}, \bar{b} \in \bar{c} + \bar{d}$ , then  $\bar{a} + \bar{b} \subseteq \bar{c} + \bar{d}$ . Note that  $S + S \subseteq S$  implies [PQ5]. Of course, we will use this description freely.

A *form* over  $q(A)$  is just an  $n$ -tuple  $\varphi = \langle \bar{a}_1, \dots, \bar{a}_n \rangle \in q(A)$ . The *isometry relation* is the relation  $\langle a, b \rangle \equiv \langle c, d \rangle$  if and only if  $ab = cd$  and  $a + b = c + d$ . The extension of  $\equiv$  to an  $n$ -ary relation is the standard construction, as in [2].

**Example 2.3**  $(\mathbb{Z}, \mathbb{Z}^2 \setminus \{0\})$  is a pq-ring. More generally, if  $D$  is a domain with  $\text{char}(D) \neq 2$ , then  $(D, D^2 \setminus \{0\})$  is a pq-ring. If  $F$  is a field with  $\text{char}(F) \neq 2$ , then  $(F, F^2 \setminus \{0\})$  is a pq-ring and if  $T \subseteq F$  is a preordering, then  $(F, T \setminus \{0\})$  is a rpq-ring.

As counterexample,  $(\mathbb{Z}_4, \mathbb{Z}_4^2 \setminus \{\bar{0}\})$  are not a pq-ring:  $\mathbb{Z}_4^2 \setminus \{\bar{0}\} = \{\bar{1}\}$  and  $\bar{1} \notin \bar{1} + \bar{1}$ .

**Definition 2.4 (Morphism)** A map  $f : (A, S) \rightarrow (B, T)$  will be a *morphism* of pq-rings if  $f : A \rightarrow B$  is an homomorphism of rings and  $f(S) \subseteq T$ . A morphism of rpq-rings is just a morphism of the pq-rings associated.

The category of (reduced) promiscuously quadratic rings will be denoted by **PQR** (**RPQR**).

**Proposition 2.5** *The correspondence  $(A, S) \mapsto q(A)$  is functorial.*

**Proposition 2.6** *The categories **PQR** and **RPQR** has products.*

**Proposition 2.7** *The categories **PQR** and **RPQR** has direct inductive limits.*

**Proposition 2.8** *The functor  $(A, S) \mapsto q(A)$  preserves all products and direct inductive limits.*

### 3 pq-Rings, special groups and realsemigroups

As immediately corollary of definition 2.2 we have:

**Lemma 3.1** *Is  $(A, S)$  is a (reduced) pq-ring then  $(A/mS)^\times$  is a (reduced) pre-special group.*

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